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Invasive competition with Fokker-Planck diffusion and noise

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Abstract

Species competition in a fluctuating environment is often modelled with stochastic reaction-diffusion equations. In most cases the movement of individuals is described as Fickian diffusion. However, in heterogeneous environments this is not the first choice. Recently, it has been shown that Fokker-Planck diffusion describes the movement of species in a more realistic way. Fickian diffusion always leads to spatially uniform stationary distributions whereas the Fokker-Planck diffusion generates nonuniform solutions according to the heterogeneity of the environment and the corresponding spatial variation of diffusion. Species accumulate in regions of low diffusivity and tend to lower their densities in areas of high diffusivity. In the present paper, the impact of Fokker-Planck diffusion is studied with particular consideration of changing spatio-temporal population patterns during the competitive invasion of a spatially heterogeneous, populated habitat. The standard Lotka-Volterra competition model is applied to describe the resident-invader interaction. The resident is assumed to be adapted to the heterogeneous living conditions, i.e., its motion is modelled as space-dependent Fokker-Planck diffusion. The invader's diffusion is taken as neutral Fickian. Furthermore, it is shown that multiplicative environmental noise can either foster or hinder the invasion.

Keywords: Competition, invasion, Fokker-Planck diffusion, environmental noise

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0 1. Introduction

1 Biological invasions are a growing threat to biodiversity around the world. The spread of
2 alien species can lead not only to the extinction of indigenous species, but also cause consid-
3 erable economic damage in eco- and agro-ecosystems. These are, contrary to water-air-soil
4 contamination, unfortunately mostly irreparable, since the invasive species constantly re-
5 produce and often have no natural enemies. Well-known examples are the invasion of
6 rabbits (Perrings et al., 2000) and cane toads (Urban et al., 2008) in Australia or the in-
7 vasion of a Pacific oyster in the Dutch and German Wadden Sea (Diederich et al., 2005).
8 Many of these invasions are caused by anthropogenic effects such as globalized traffic and
9 trade or climate change. On the other hand it might be of interest to reintroduce a species
10 into an ecosystem where it has been eradicated before.

11 Modelling the invasion process can help to identify mechanisms which foster or hinder
12 successful invasions. Most ecological models on this subject describe species densities on
13 a population level. Individual movement is often described as diffusive process. In a
14 heterogeneous environment the mathematical formalism which describes this process in a
15 realistic way depends strongly on the respective species abilities.

16 In a recent publication (Bengfort et al., 2016), different formulations for diffusivities have
17 been assumed purely space-dependent. Spatial patterns may already occur without any
18 interactions. For this setting, the spatially stationary solution has been derived. Fur-
19 thermore, the speed of diffusive waves of a single logistically growing population has been
20 analytically estimated, and conditions for the formation of spatio-temporal and Turing
21 patterns in an excitable prey-predator system have been given.

22 Another recent publication (Siekmann and Malchow, 2016) has dealt with the control of
23 invasion of a populated habitat by selective infection of the invader. Furthermore, the area
24 is subject to uncorrelated and/or correlated environmental noise the resident is adapted to
25 but the invading population not.

26 The present work shall link the two latter approaches. The Lotka-Volterra textbook model
27 of the competition of two populations is combined with space-dependent Fokker-Planck
28 diffusion of the residents, Fickian diffusion of the invaders and environmental noise. Two
29 competing species, X_1 and X_2 , are modelled with equal mutual competition rates but
30 different movement abilities. It is assumed that the resident species is adapted to the
31 environmental conditions in its habitat. Favourable regions are characterized by a lower

32 diffusion coefficient whereas it is rather high in unfavourable patches. Hence, the resident
 33 species move fast in unfavourable regions and tend to remain in the favoured. On the
 34 other hand, the invader's diffusivity is spatially uniform, i.e., it moves independent of the
 35 environmental conditions. It will be shown that the spatial heterogeneity modelled by
 36 Fokker-Planck diffusion but also the external noise can foster or hinder the invasion.

37 2. Methods

38 Interactions and movements of populations in a heterogeneous and variable environment
 39 are often modelled with stochastic reaction-diffusion equations. Diffusive fluxes in ecology
 40 can differ due to specifics of the population's relationships and environmental heterogeneity.
 41 They might be neutral cf. eq. (1), attractive (2) or repulsive (3), i.e., for N populations

$$\vec{j}_{in} = -D_i(\vec{r}, \mathbf{X}) \vec{\nabla} X_i(\vec{r}, t), \quad (1)$$

$$\vec{j}_{ia} = -D_i^2(\vec{r}, \mathbf{X}) \vec{\nabla} \left[\frac{X_i(\vec{r}, t)}{D_i(\vec{r}, \mathbf{X})} \right] = +X_i(\vec{r}, t) \vec{\nabla} D_i(\vec{r}, \mathbf{X}) - D_i(\vec{r}, \mathbf{X}) \vec{\nabla} X_i(\vec{r}, t), \quad (2)$$

$$\vec{j}_{ir} = -\vec{\nabla} [D_i(\vec{r}, \mathbf{X}) X_i(\vec{r}, t)] = -X_i(\vec{r}, t) \vec{\nabla} D_i(\vec{r}, \mathbf{X}) - D_i(\vec{r}, \mathbf{X}) \vec{\nabla} X_i(\vec{r}, t); \quad (3)$$

$$i = 1, 2, \dots, N.$$

42 The usual notation is used: $\mathbf{X}(\vec{r}, t) = \{X_i(\vec{r}, t); i = 1, 2, \dots, N\}$ is the vector of population
 43 densities at position $\vec{r} = \{x, y\}$ and time t and $D_i(\vec{r}, \mathbf{X})$ their possibly space- and density-
 44 dependent diffusion coefficient. The formulations (1–3) have been elaborated by Skellam
 45 (1951; 1973, and nicely summarized by Okubo (1980), see also Aronson (1985) and Murray
 46 (1989). In order to complete the list of ecodiffusive fluxes in heterogeneous media, one could
 47 add the flux in environmental potentials $U(\vec{r})$

$$\vec{j}_{ip} = \vec{j}_{ik} + \gamma_i X_i(\vec{r}, t) \vec{\nabla} U(\vec{r}); \quad i = 1, 2, \dots, N; \quad (4)$$

48 where γ_i is called the coefficient of affinity of X_i to the environment and index k can
 49 be n , a and r respectively, i.e., one of the fluxes (1–3) can be applied. The minima of
 50 $U(\vec{r})$ correspond to preferable and, therefore, attracting habitats. The latter concept has
 51 been derived from the ideas of habitat value and environmental density (Morisita, 1971;
 52 Shigesada and Teramoto, 1978).

53 The neutral diffusion is also called Fickian (Fick, 1855) whereas the repulsive type is named
 54 after Fokker and Planck (1914; 1917). For a certain density dependence of diffusion, the

55 latter has been used for modelling the spatial segregation of populations (Shigesada et al.,
56 1979; Mimura and Kawasaki, 1980) as well as the formation of Turing patterns (Malchow,
57 1988).

58 *2.1. The stochastic competition-diffusion model*

59 The dynamics of resident X_1 and invader X_2 is described by

$$\frac{\partial X_1}{\partial t} = (1 - X_1)X_1 - c_{12}X_1X_2 + d_1\nabla^2(X_1D^*(x, y)) + g_1(X_1)\xi(\vec{r}, t), \quad (5)$$

$$\frac{\partial X_2}{\partial t} = (1 - X_2)X_2 - c_{21}X_1X_2 + d_2\nabla^2X_2 + g_2(X_2)\xi(\vec{r}, t). \quad (6)$$

60 c_{12} and c_{21} are parameters to describe the strength of interspecific competition between
61 X_1 and X_2 . d_1 and d_2 are constant parameters to describe the strength of diffusion. Both
62 species are assumed to grow logistically. The space dependence of the resident's diffusivity
63 is chosen as

$$D^*(x, y) = D_0 + \begin{cases} a \left(\sin(\sqrt{x^2 + y^2}) \right)^m & \text{if } \sqrt{x^2 + y^2} < 3\pi, \\ a (\sin(3\pi))^m & \text{else.} \end{cases} \quad (7)$$

64 This spatially varying diffusivity is meant to represent a simple fragmented landscape with
65 a varying habitat quality for species X_1 . The parameter m is an even number which controls
66 the steepness of D^* . Zero-flux boundary conditions are assumed. In order to avoid effects
67 from spatial heterogeneities at the boundaries, D^* is set constant for $\sqrt{x^2 + y^2} < 3\pi$,
68 whereas $x = 0, y = 0$ defines the center of the landscape.

69 For simplicity, just uncorrelated white noise $\xi(\vec{r}, t)$ is applied here, i.e.,

$$\langle \xi(\vec{r}, t) \rangle = 0, \langle \xi(\vec{r}_1, t_1)\xi(\vec{r}_2, t_2) \rangle = \delta(\vec{r}_1 - \vec{r}_2)\delta(t_1 - t_2) \quad (8)$$

70 with linearly density dependent noise intensities

$$g_i(X_i) = \omega_i X_i; i = 1, 2. \quad (9)$$

71 *2.2. Numerical methods*

72 We integrate the equations (5) and (6) numerically with a splitted scheme. Therefor we
73 use a different method to solve the diffusion terms than we use for the reaction and noise
74 terms on the right hand side of the equations. Both numerical methods are explained in the
75 following.

76 *2.2.1. Crank-Nicolson scheme for two dimensions with Fokker-Planck diffusion*

77 We split the Laplace operator into two parts. First, we calculate the diffusion in one spatial
78 dimension (x), second we do the same for the other spatial dimension (y).

$$\frac{\partial X}{\partial t} = \vec{\nabla}^2(XD) = \frac{\partial^2(XD)}{\partial x^2} + \frac{\partial^2(XD)}{\partial y^2}, \quad (10)$$

79 where X is the population density and D its spatially varying diffusion coefficient which
80 can be written as

$$D(x, y) = d_1 D^*(x, y) \quad (11)$$

81 with $d_1 = \text{const}$ and $D^*(x, y) \neq 0 \forall x, y$. Now we formulate the Crank-Nicolson algorithm
82 (Crank and Nicolson, 1947) for one spatial dimension as follows

$$\begin{aligned} \frac{X_k^{t+\Delta t} - X_k^t}{\Delta t} = \frac{d_1}{2\Delta x^2} & \left(X_{k+1}^{t+\Delta t} D_{k+1}^* - 2X_k^{t+\Delta t} D_k^* + X_{k-1}^{t+\Delta t} D_{k-1}^* \right. \\ & \left. + X_{k+1}^t D_{k+1}^* - 2X_k^t D_k^* + X_{k-1}^t D_{k-1}^* \right). \end{aligned} \quad (12)$$

83 Here $k \in (1, n)$ is the index of the spatial position of X , whereas t is the time which varies
84 with a discrete time step Δt . With $\alpha = d_1 \frac{\Delta t}{\Delta x^2}$ we can write this as a system of linear
85 equations

$$\mathbf{A} \left(\vec{X}^{t+\Delta t} \vec{D}^* \right) = \mathbf{B} \left(\vec{X}^t \vec{D}^* \right) \quad (13)$$

86 where \vec{X} and \vec{D}^* are vectors of length n including the values of X_k and D_k^* at each spatial
87 position in one dimension $k \in (1, n)$. \mathbf{A} and \mathbf{B} are the $(n \times n)$ tridiagonal matrices

$$\mathbf{A} = \begin{pmatrix} 2 \left(\frac{1}{D_1^*} + \alpha \right) & -\alpha & 0 & \dots & 0 \\ -\alpha & \ddots & -\alpha & 0 & \vdots \\ 0 & -\alpha & \ddots & \ddots & 0 \\ \vdots & \dots & \ddots & \ddots & -\alpha \\ 0 & \dots & -\alpha & 2 \left(\frac{1}{D_n^*} + \alpha \right) \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 2 \left(\frac{1}{D_1^*} - \alpha \right) & \alpha & 0 & \dots & 0 \\ \alpha & \ddots & \alpha & 0 & \vdots \\ 0 & \alpha & \ddots & \ddots & 0 \\ \vdots & \dots & \ddots & \ddots & \alpha \\ 0 & \dots & \alpha & 2 \left(\frac{1}{D_n^*} - \alpha \right) \end{pmatrix}.$$

88 This implicit scheme has been proven to be unconditionally stable for two spatial di-
 89 mensions in case of homogeneous diffusion. A strictly positive D^* does not change the
 90 von-Neumann stability criterion given by Crank and Nicolson (1947).

91 In order to implement zero-flux boundary conditions we have to add the term $-\alpha$ to the
 92 matrix components \mathbf{A}_{11} and \mathbf{A}_{nn} , and the term α to the matrix components \mathbf{B}_{11} and \mathbf{B}_{nn} .
 93 To calculate the distribution of X^t at time step $t + \Delta t$, we have to multiply the vector \vec{X}^t
 94 with the spatially varying coefficient of diffusion \vec{D}^* and solve the equation $\mathbf{A}\vec{Y} = \mathbf{B}\vec{X}$,
 95 where \vec{X} is a input-vector (in our case $\vec{X}^t \cdot \vec{D}^*$) and \vec{Y} is a output-vector. After that the
 96 components of the output-vector \vec{Y} has to be divided with the corresponding components
 97 of the vector \vec{D}^* , which is temporally constant in order to get the distribution $X^{t+\Delta t}$. Once
 98 this scheme has been performed for each row in one spatial direction it has to be repeated
 99 for the other spatial dimension in every time step.

100 2.2.2. Derivative-free Milstein method for interactions and noise

101 For numerical integration of the interaction and noise terms, the derivative-free Milstein
 102 method is used (Milstein, 1995; Kloeden and Platen, 1999). The Milstein scheme reads for
 103 white noise (8,9) with time step Δt and in Stratonovich interpretation

$$X_i^{t+\Delta t} = X_i^t + f_i(X_i^t)\Delta t + \omega_i X_i^t \Delta W_i + \frac{\omega_i}{2} \left[f_i(X_i^t) \sqrt{\Delta t} + \omega_i X_i^t \right] (\Delta W_i)^2, \quad (14)$$

with

$$\Delta W_i = W_i^{t+\Delta t} - W_i^t \sim \sqrt{\Delta t} \mathcal{N}(0, 1),$$

104 where W_i^t is a Wiener process and $\mathcal{N}(0, 1)$ stands for the normal distribution with zero mean
 105 and unity variance. The required uniformly distributed random numbers are generated
 106 with the Mersenne Twister (Matsumoto and Nishimura, 1998), the normally distributed
 107 random numbers with the common Box-Muller algorithm (Box and Muller, 1958). More
 108 details about this scheme can be found in Siekmann and Malchow (2016).

109 3. Numerical simulations and results

110 The model as well as the algorithms described in this paper were implemented in FORTRAN.
 111 The above mentioned numerical schemes are performed successively for each time step.
 112 The dimensionless spatial model describes a rectangular domain with a length of $L_x =$

113 $L_y = 3000$ with 200×200 grid points and zero-flux boundary conditions. The temporal
114 and spatial step sizes h_t , h_x and h_y were

$$h_t = 0.02, h_x = h_y = 15. \quad (15)$$

115 The following parameters have been applied:

$$D_0 = 1, m = 8, c_{12} = c_{21} = 1.2.$$

116 The parameter values are chosen arbitrarily without a limitation of generality. We set
117 $D_0 = 1$ because this is the minimal value for D^* . In this case the diffusivity of both species
118 can directly be spotted from d_1 and d_2 . The effect of Fokker-Planck diffusion depends on
119 the heterogeneity of the coefficient of diffusion. We set $m = 8$ in order to generate large
120 values in the first and second derivative of D^* . Because both species are described with
121 equal competition parameters (c_{12} and c_{21}), the difference in the coefficient of diffusion
122 determines whether or not an invasion of species X_2 is successful in case of homogeneous
123 D , i.e., $a = 0$, cf. eq. (7). Both c_{12} and c_{21} exceed unity, so that the system is in a bistable
124 parameter range. In case of a non-spatial model without diffusion or noise the species with
125 a larger initial density will become dominant and drive its opponent to extinction.

126 Due to its spatial variation, the diffusivity $d_1 \cdot D^*$ of the native species X_1 can be greater
127 or less than the constant invader's diffusion coefficient d_2 . An invasion will be successful
128 only in those areas where $d_2 > d_1 \cdot D^*$ (Fig. 2). Areas with a high diffusivity of the native
129 species act as barrier for the invasion. This fits well earlier published results on diffusion-
130 controlled competitive invasions (Malchow et al., 2011). In this scenario multiplicative
131 density-dependent noise (8,9) accelerates the speed of invasion (Fig. 2b). However, strong
132 noise can push the invader through the barriers of large resident diffusivity and induce the
133 invasion of further regions with low resident diffusivity.

134 Because of the Fokker-Planck diffusion in eq. (5), the spatial distribution of the resident
135 species, X_1 , develops proportional to $\nabla^2 D^*(x, y)$, as described in Bengfort et al. (2016). If
136 this effect is strong enough, the reduced resident concentration in areas with high resident
137 diffusivity enables an invasion of species X_2 , even if the diffusivity of X_1 is larger than
138 the diffusivity of X_2 everywhere in the domain (Fig. 3). In this scenario, multiplicative
139 density-dependent noise has a decelerating effect on the speed of invasion (Fig. 3b).

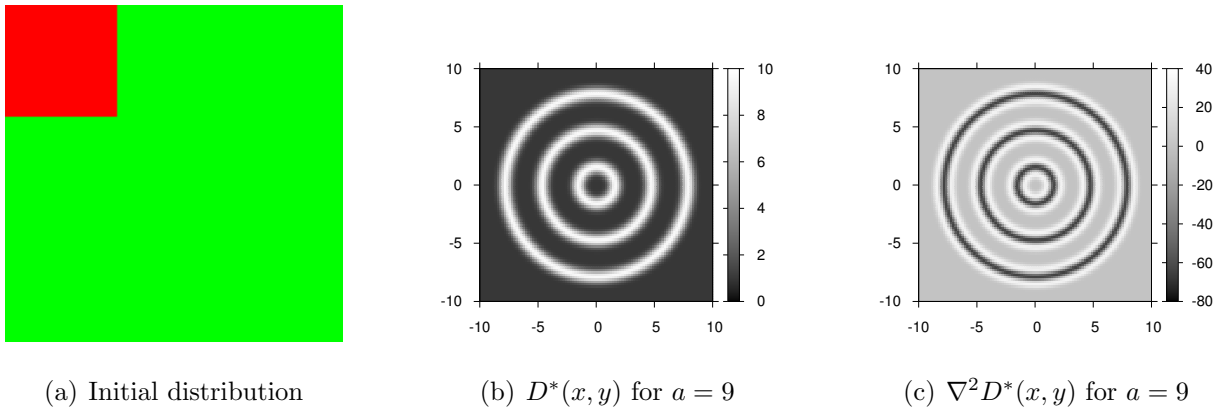
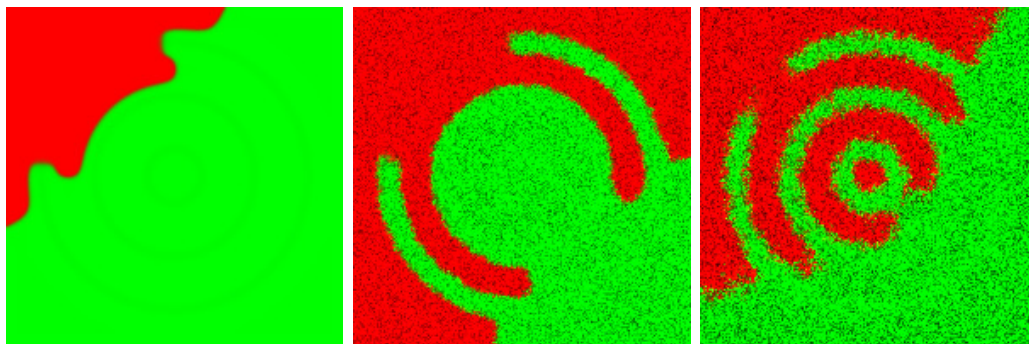
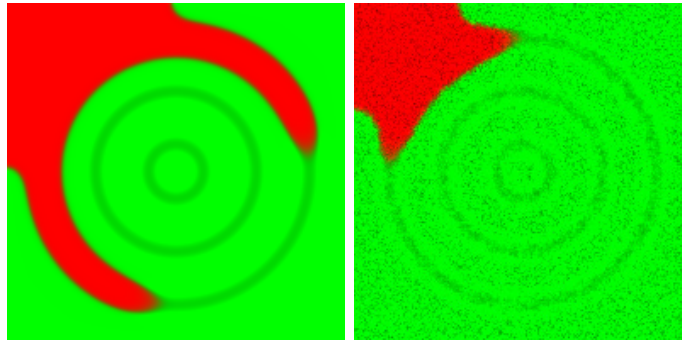


Figure 1: Initial settings for densities (green = resident, red = invader) and resident's diffusivity



(a) $t = 3900; \omega_1 = \omega_2 = 0$ (b) $t = 3900; \omega_1 = \omega_2 = 0.4$ (c) $t = 600; \omega_1 = \omega_2 = 0.6$

Figure 2: $d_1 = 5, d_2 = 25, a = 9$: The density of the resident species is reduced in areas of large D^* . The invader successfully invades the space, where it has a larger coefficient of diffusion as the resident species. Density-dependent multiplicative noise accelerates the invasion in areas of small D^* . Areas with large D^* act as a barrier for the invasion. Strong noise can break through these barriers and induces invasion of X_2 in the inner circles with small D^* . Video sequences showing the dynamical process are included in the online version of this document. (Click on the image.)



(a) $\omega_1 = \omega_2 = 0$

(b) $\omega_1 = \omega_2 = 0.25$

Figure 3: $t = 4400$, $d_1 = 30$, $d_2 = 25$, $a = 19$; large $\nabla^2 D(\vec{r})$: Due to the reduced resident concentration in areas of large D^* invasion is possible even though the invader has a smaller coefficient of diffusion everywhere in the spatial domain. Noise reduces the invasion speed. Strong noise can invert the invasion. Video sequences showing the dynamical process are included in the online version of this document. (Click on the image.)

140 4. Discussion

141 This Fokker-Planck type modelling of the movement of organisms generates patterns in the
142 spatial population distribution which correspond to the spatial variation of the diffusion
143 coefficient. If this effect is small (small spatial derivatives in D^*), the competitor can invade
144 the domain in areas where its (spatially constant) coefficient of diffusion is larger than that
145 of the resident species. This is not surprising because both species are described with equal
146 parameters for growth and competition so that diffusivity determines the success of invasion
147 if the size of the initial patch of the invading species exceeds the related critical patch size.
148 This is also the case if the resident would follow Fickian diffusion with a heterogeneous
149 coefficient of diffusion. In a non-deterministic environment, where the populations are
150 subject to stochastic fluctuations, the speed of invasion increases with increasing noise
151 intensity. Strong noise can also induce invasions in areas which are perfectly protected
152 against an invasion in the deterministic case. If the pattern forming effect of the Fokker-
153 Planck diffusion is stronger, invasion is possible even though the coefficient of the invader
154 is smaller than the one of the resident species everywhere in the domain. Contrary to the
155 former example, noise has a negative effect on the success of invasion. This is caused by
156 the fact, that the density dependent noise counteract the pattern forming properties of the
157 Fokker-Planck diffusion. The resident species benefits from the homogenising effect of the
158 noise because it has a larger coefficient of diffusion than the invader. A Fickian diffusion for
159 the resident can not reproduce these patterns. In this case the resident would outcompete
160 the invader because of its larger coefficient of diffusion.

161 5. Conclusions

162 It has been shown that a non-uniform diffusivity, i.e., Fokker-Planck diffusion, of a resident
163 species in a spatially heterogeneous habitat can have different effects on the ability of a
164 similar competing species to invade the habitat.

165 Spatiotemporal Gaussian noise was applied in order to model the variability of the envi-
166 ronment. For future research it would be interesting to investigate the effect of spatially
167 and/or temporally colored noise in combination with the Fokker-Planck diffusion which
168 generates patterns in the resident species with a certain wavelength.

169 Here, it was assumed that only the resident species favours certain areas in the domain and
170 consequently move with a spatially varying speed and is therefore described with Fokker-

171 Planck diffusion. One can also think of an inverse situation where the resident species
172 is described with constant Fickian diffusion and the invader follows the Fokker-Planck
173 description. From a theoretical point of view this scenario is not as interesting because the
174 invader starts in a relatively small spatial domain where the heterogeneity in the coefficient
175 of diffusion does not play a crucial role. A situation in which both species follow a Fokker-
176 Planck description and favour the same or different spatial domains was not part of this
177 study. This will be subject of future investigations.

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