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On Energy and Entropy Influxes in the Green–Naghdi Type III Theory of Heat Conduction

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The energy-influx / entropy-influx relation in the Green–Naghdi Type III theory of heat conduction is examined within a thermodynamical framework *à la* Müller–Liu, where that relation is not specified a priori irrespectively of the constitutive class under attention. It is shown that the classical assumption, i.e., that the entropy influx and the energy influx are proportional via the absolute temperature, holds true if heat conduction is, in a sense that is made precise, isotropic. In addition, it is proven that the standard assumption cannot be postulated in general in case of transversely isotropic conduction.

Key words: energy influx, entropy influx, Müller–Liu entropy principle, coldness, thermal displacement.

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1. Introduction

A key assumption in the standard ontinuum theory of heat propagation is that, for $\mathbf{q},\mathbf{h}$ and r,s the influx vectors and the external sources of, respectively, energy and entropy, the energy inflow $(\mathbf{q},r)$ is proportional to the entropy inflow $(\mathbf{h},s)$, the proportionality factor being the absolute temperature ϑ:

$$(\mathbf{q},r)=\vartheta(\mathbf{h},s),\quad \vartheta>0. \tag{1.1}$$

Whether or not this assumption is generally tenable has been the subject of considerable debate (see e.g. the papers by Müller (1971) and Liu (1972, 1973)). The issue can be taken up in the framework of one or another continuum theory of heat conduction: Podio–Guidugli (2012) uses the framework of the standard theory, which leads to a parabolic heat equation, Bargmann & Steinmann (2007) that of the Type III theory of Green & Naghdi (1993), which allows for propagation of heat waves with finite speed (we summarize these two theories in Section 2). In this paper, we use the latter framework, in a manner that differs from Bargmann and Steinmann's in a number of points. As in Podio–Guidugli (2012) for the standard theory, we prove: in Section 3, that for thermodynamic consistency (1.1) must hold true when heat conduction is, in a sense that we make precise, *isotropic*; in Section 4, that (1.1) may be violated in the *transversely isotropic* case. In our proofs, we exploit certain simple mathematical tools that are described in Appendices A and B.

2. Heat conduction theories

Thermal and deformational phenomena can be considered coupled, as is done, e.g., in thermoelasticity. For our present purposes, it is sufficient to consider the simplest instance, namely, heat propagation in a rigid conductor. Consequently, we need not distinguish between reference and current configurations; we also leave all spatial dependences tacit.

The standard continuum theory of heat conduction is based on the following two laws:
– (*energy balance*)

$$\dot{\varepsilon} = -\mathrm{div}\mathbf{q} + r, \tag{2.1}$$

where ε denotes the volume density of the internal energy, and $\dot{\varepsilon}$ its time rate;
– (*entropy imbalance*)

$$\dot{\eta} \geq -\mathrm{div}\mathbf{h} + s, \tag{2.2}$$

with η the entropy. It is customarily assumed that (1.1) holds. With this and the notion of *Helmholtz free-energy*:

$$\psi = \varepsilon - \vartheta\eta,$$

the following *free-energy growth inequality* is arrived at:

$$\dot{\psi} + \eta\dot{\vartheta} + \vartheta^{-1}\,\mathbf{q}\cdot\nabla\vartheta \leq 0\,. \tag{2.3}$$

When a set of state variables is chosen – say, $\{\vartheta, \nabla\vartheta, \dot{\vartheta}\}$ – this inequality is used to derive restrictions on the choice of the state functions delivering ψ, η, and $\mathbf{q}$, in the manner devised by Coleman and Noll in their classic paper (Colemann & Noll 1963).

Green & Naghdi (1991) modified the classic path, in that they took the following entropy law as their point of departure:
– (*entropy balance*)

$$\xi = \dot{\eta} + \mathrm{div}\mathbf{h} - s, \tag{2.4}$$

and they chose not to assume that the *internal entropy production* ξ is nonnegative. As to internal energy, they accepted the balance equation (2.1), implicitly excluding any internal production of energy; as to entropy and energy inflows, the classic proportionality expressed by (1.1). Consequently, they arrived at the following version of (2.4) in terms of Helmholtz free energy:

$$\dot{\psi} + \eta\dot{\vartheta} + \mathbf{h}\cdot\nabla\vartheta + \xi = 0. \tag{2.5}$$

Moreover, in their Type III theory, they assumed that the constitutive mappings delivering ψ, η, and $\mathbf{h}$, depend on the following list of state variables:

$$\mathcal{S} := \{\alpha, \dot{\alpha}, \nabla\alpha, \nabla\dot{\alpha}\}, \tag{2.6}$$

where the *thermal displacement* α, a field whose precise interpretation in statistical mechanics is still wanted, is somehow indirectly characterized by postulating that its time derivative equals an *empirical temperature* T:

$$\dot\alpha(t) := T(t).$$[1]

As anticipated in the Introduction, Müller and Liu questioned the general applicability of assumption (1.1) and proposed a thermodynamic format within which to test whether or not it fits a given constitutive class. This format amounts to considering the energy balance (2.1), with $r=0$, as an internal constraint to be appended to the entropy imbalance (2.2), with $s=0$, after multiplication by a so-called 'Lagrange multiplier' λ, a positive scalar to be constitutively specified:

$$\dot\eta + \mathrm{div}\mathbf{h} - \lambda\big(\dot\varepsilon + \mathrm{div}\mathbf{q}\big) \geq 0, \qquad \lambda > 0. \tag{2.7}$$

When the constitutive mappings involved in (2.7) depend on the Green–Naghdi list (2.6) of state variables, one can try and use a Coleman–Noll type procedure to see whether the multiplier λ may be shown to be, if not equal, at least proportional to *coldness* (Müller 1971), that is, the inverse ϑ^{-1} of the absolute temperature.

Granted an appropriate generalization of the format we just introduced, the same issue has been considered by Bargmann & Steinmann (2007) for thermoelastic materials. We here reach more complete conclusions than theirs under less stringent assumptions and by a different train of reasoning.

3. Isotropic conduction implies proportionality of energy and entropy influxes

On choosing the state variables as in (2.6), inequality (2.7) reads:

$$\begin{aligned}&\big(\partial_\alpha\eta - \lambda\,\partial_\alpha\varepsilon\big)\dot\alpha + \big(\partial_{\dot\alpha}\eta - \lambda\,\partial_{\dot\alpha}\varepsilon\big)\ddot\alpha + \big(\partial_{\nabla\alpha}\eta - \lambda\,\partial_{\nabla\alpha}\varepsilon\big)\cdot\nabla\dot\alpha + \\ &\big(\partial_{\nabla\dot\alpha}\eta - \lambda\,\partial_{\nabla\dot\alpha}\varepsilon\big)\nabla\ddot\alpha + \mathrm{div}\mathbf{h} - \lambda\,\mathrm{div}\mathbf{q} \geq 0,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}\mathrm{div}\mathbf{h} - \lambda\mathrm{div}\mathbf{q} = &\big(\partial_\alpha\mathbf{h} - \lambda\,\partial_\alpha\mathbf{q}\big)\cdot\nabla\alpha + \big(\partial_{\dot\alpha}\mathbf{h} - \lambda\,\partial_{\dot\alpha}\mathbf{q}\big)\cdot\nabla\dot\alpha + \\ &+ \big(\partial_{\nabla\alpha}\mathbf{h} - \lambda\,\partial_{\nabla\alpha}\mathbf{q}\big)\cdot\nabla^2\alpha + \big(\partial_{\nabla\dot\alpha}\mathbf{h} - \lambda\,\partial_{\nabla\dot\alpha}\mathbf{q}\big)\cdot\nabla^2\dot\alpha.\end{aligned} \tag{3.2}$$

We apply the Coleman–Noll procedure (Coleman & Noll 1963) to the inequality that is obtained by combining (3.1) and (3.2), that is, we require that it

---

[1] Here is a quotation from Green & Nagdi (1991): "The temperature T (on the macroscopic scale) is generally regarded as representing (on thermolecular scale) some 'mean' velocity magnitude or 'mean' (kinetic energy)$^{1/2}$. With this in mind, we introduce a scalar $\alpha = \alpha(X,t)$ through an integral of the form

$$\alpha = \int_{t_0}^{t} T(X,\tau)\,d\tau + \alpha_0, \tag{7.3}$$

where t_0 denotes some reference time and the constant α_0 is the initial value of α at time t_0. In view of the above interpretation associated with T and the physical dimension of the quantity defined by (7.3), the variable α may justifiably be called thermal displacement magnitude or simply thermal displacement. Alternatively, we may regard the scalar α (on the macroscopic scale) as representing a 'mean' displacement magnitude on the molecular scale and then $T = \dot\alpha$."

be satisfied whatever the local continuation $\{\ddot\alpha, \nabla\ddot\alpha, \nabla^2\alpha, \nabla^2\dot\alpha\}$ of each admissible process. This requirement is satisfied if and only if

$$
\begin{aligned}
&\mathrm{sym}\big(\partial_{\nabla\alpha}\mathbf{h} - \lambda\,\partial_{\nabla\alpha}\mathbf{q}\big) = \mathbf{0},\\
&\mathrm{sym}\big(\partial_{\nabla\dot\alpha}\mathbf{h} - \lambda\,\partial_{\nabla\dot\alpha}\mathbf{q}\big) = \mathbf{0},\\
&\partial_{\dot\alpha}\eta - \lambda\,\partial_{\dot\alpha}\varepsilon = 0,\\
&\partial_{\nabla\dot\alpha}\eta - \lambda\,\partial_{\nabla\dot\alpha}\varepsilon = 0,
\end{aligned}
\tag{3.3}
$$

and, moreover, inequality (2.7) reduces to

$$
\begin{aligned}
&\big(\partial_\alpha\eta - \lambda\,\partial_\alpha\varepsilon\big)\dot\alpha + \big(\partial_{\nabla\alpha}\eta - \lambda\,\partial_{\nabla\alpha}\varepsilon + \partial_{\dot\alpha}\mathbf{h} - \lambda\,\partial_{\dot\alpha}\mathbf{q}\big)\cdot\nabla\dot\alpha +\\
&\big(\partial_\alpha\mathbf{h} - \lambda\,\partial_\alpha\mathbf{q}\big)\cdot\nabla\alpha \geq 0.
\end{aligned}
\tag{3.4}
$$

Conditions $(3.3)_{1,2}$ are expedient to prove our main result:

For each chosen value of the independent variables $\alpha, \dot\alpha$, let the energy and entropy influxes and the Lagrangian multiplier be delivered by isotropic *constitutive mappings $\widehat{\mathbf{q}}(\alpha,\dot\alpha,\cdot,\cdot)$, $\widehat{\mathbf{h}}(\alpha,\dot\alpha,\cdot,\cdot)$, $\widehat\lambda(\alpha,\dot\alpha,\cdot,\cdot)$, the first two vector-valued, the third delivering positive scalars. Then, the energy influx is proportional to the entropy influx via the Lagrangian multiplier, which depends neither on the thermal displacement gradient $\nabla\alpha$ nor on the temperature gradient $\nabla\dot\alpha$:*

$$
\mathbf{h} = \lambda\,\mathbf{q}, \quad \lambda = \widehat\lambda(\alpha,\dot\alpha) > 0. \tag{3.5}
$$

Our proof is achieved as follows. For simplicity, we let the dependence on α and $\dot\alpha$ be tacit, and write $\mathbf{u}$ for the third state variable in the list (2.6), and $\mathbf{v}$ for the fourth. On making use of the representation formula (5.3) from Appendix A, we set:

$$
\begin{aligned}
&\widehat{\mathbf{h}}(\mathbf{u},\mathbf{v}) = h_1\mathbf{u} + h_2\mathbf{v}, \quad h_j = \widehat{h}_j(|\mathbf{u}|,|\mathbf{v}|,\mathbf{u}\cdot\mathbf{v})\ (j=1,2),\\
&\widehat{\mathbf{q}}(\mathbf{u},\mathbf{v}) = q_1\mathbf{u} + q_2\mathbf{v}, \quad q_j = \widehat{q}_j(|\mathbf{u}|,|\mathbf{v}|,\mathbf{u}\cdot\mathbf{v})\ (j=1,2),
\end{aligned}
\tag{3.6}
$$

whence

$$
\mathbf{h} - \lambda\,\mathbf{q} = (h_1 - \lambda\,q_1)\mathbf{u} + (h_2 - \lambda\,q_2)\mathbf{v}; \tag{3.7}
$$

provisionally, we also set

$$
\lambda = \overline{\lambda}(|\mathbf{u}|,|\mathbf{v}|,\mathbf{u}\cdot\mathbf{v}).
$$

With this, condition $(3.3)_1$ reads:

$$
\begin{aligned}
&(h_1 - \lambda q_1)\mathbf{I} + \big(\partial_1 h_1 - \lambda\partial_1 q_1\big)|\mathbf{u}|^{-1}\,\mathbf{u}\otimes\mathbf{u} + \big(\partial_3 h_2 - \lambda\partial_3 q_2\big)\,\mathbf{v}\otimes\mathbf{v} +\\
&\Big(\partial_3 h_1 + \partial_1 h_2|\mathbf{u}|^{-1} - \lambda\big(\partial_3 q_1 + \partial_1 q_2|\mathbf{u}|^{-1}\big)\Big)\,\mathrm{sym}\big(\mathbf{u}\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}\big) = \mathbf{0},
\end{aligned}
\tag{3.8}
$$

and has to be satisfied for all $\mathbf{u}$ and $\mathbf{v}$ (here ∂_i $(i=1,2,3)$ denotes differentiation of a function with respect to the i-th of its arguments). An application of the first of the two algebraic lemmas proved in Appendix B permits us to conclude that:

$$\begin{aligned} &h_1 - \lambda q_1 = 0, \\ &\partial_1\big(h_1 - \lambda q_1\big) + \big(\partial_1\lambda\big) q_1 = 0, \\ &\partial_3 h_2 - \lambda\,\partial_3 q_2 = 0, \\ &\partial_3 h_1 - \lambda\,\partial_3 q_1 + (\partial_1 h_2 - \lambda\,\partial_1 q_2)|\mathbf{u}|^{-1} = 0. \end{aligned} \tag{3.9}$$

Quite analogously, condition $(3.3)_2$ yields:

$$\begin{aligned} &h_2 - \lambda q_2 = 0, \\ &\partial_2\big(h_2 - \lambda q_2\big) + \big(\partial_2\lambda\big) q_2 = 0, \\ &\partial_3 h_1 - \lambda\,\partial_3 q_1 = 0, \\ &\partial_3 h_2 - \lambda\,\partial_3 q_2 + (\partial_2 h_2 - \lambda\,\partial_2 q_2)|\mathbf{v}|^{-1} = 0. \end{aligned} \tag{3.10}$$

Now, $(3.9)_{1,2}$ and $(3.10)_{1,2}$ imply that the Lagrange multiplier λ cannot depend on $|\mathbf{u}|, |\mathbf{v}|$; moreover, $(3.9)_3$ and $(3.10)_1$ allow to conclude that it does not depend on $\mathbf{u}\cdot\mathbf{v}$ as well, which establishes the second of (3.5). Consequently, again with the use of $(3.9)_1$ and $(3.10)_1$, we arrive at the following precise statement of $(3.5)_1$:

$$\widehat{\mathbf{h}}(\alpha, \dot\alpha, \nabla\alpha, \nabla\dot\alpha) = \widehat{\lambda}(\alpha, \dot\alpha)\, \widehat{\mathbf{q}}(\alpha, \dot\alpha, \nabla\alpha, \nabla\dot\alpha). \tag{3.11}$$

A direct consequence of (3.11) is that, when two material bodies are in *ideal thermal contact*, that is, by definition, when neither the energy nor the entropy flux suffers a jump at a point of a common interface oriented by the normal field $\mathbf{n}$:

$$[\![\mathbf{q}\cdot\mathbf{n}]\!] = [\![\mathbf{h}\cdot\mathbf{n}]\!] = 0, \tag{3.12}$$

then the Lagrange multiplier is also continuous at the interface:

$$[\![\lambda]\!] = 0. \tag{3.13}$$

At this point, in the words of Müller, "...there is a price to pay for not having introduced the temperature so far in rational thermodynamics with Lagrange multipliers" (Müller 1985, p. 168). The *desideratum* is a proof that

$$\lambda \propto \vartheta^{-1}, \tag{3.14}$$

with ϑ^{-1} the *coldness*; hence, in particular, that $\widehat{\lambda}$ is a universal function. This can be achieved, as Müller himself proposed in (Müller 1971), by considering a situation when a material body of whatsoever constitutive nature is put in ideal thermal contact with an ideal gas, for which the kinetic theory permits to show that (3.14) indeed holds, provided ϑ is taken proportional to the empirical temperature T. We leave it to the reader to decide whether or not such an argument is convincing. For us, in that it relies on importing a result from a theory tacitly regarded as more foundational in nature, it is for sure suggestive of assuming (3.14) right away, without any need to imply, let alone accept, a vassalage between theories.

4. A counterexample to proportionality: transversely isotropic conduction

In this Section, we show that assumption (1.1) is not tenable in general. Within the framework of a standard theory of heat conduction modified by the introduction of a Lagrangian multiplier, this result has been announced by Liu (2009), who proposed to consider the case of transversely isotropic materials to exhibit a counterexample to proportionality of energy and entropy influxes; the proof he offered is faulty, although easily amendable (Podio–Guidugli 2012). We here exploit the same counterexample within the Green–Naghdi theory of Type III.

For $\mathbf{e}\in\mathcal{V}$ any chosen unit vector, consider the class of vector-valued mappings over $\mathcal{V}\times\mathcal{V}$ of the following form:

$$\widetilde{\mathbf{f}}(\mathbf{u},\mathbf{v};\mathbf{e})=\varphi_1\,\mathbf{u}+\varphi_2\,\mathbf{v}+\varphi_3\,\mathbf{e},\quad \varphi_i=\widetilde{\varphi}_i(|\mathbf{u}|,|\mathbf{v}|,\mathbf{u}\cdot\mathbf{v},\mathbf{u}\cdot\mathbf{e},\mathbf{v}\cdot\mathbf{e})\;(i=1,2,3) \tag{4.1}$$

(once again, we have left the dependence on α and $\dot\alpha$ tacit). Clearly, each mapping $\widetilde{\mathbf{f}}$ in this class is *transversely isotropic* with respect to the axis $\mathbf{e}$, in the sense that it satisfies

$$\mathbf{Q}\widetilde{\mathbf{f}}(\mathbf{u},\mathbf{v})=\widetilde{\mathbf{f}}(\mathbf{Q}\mathbf{u},\mathbf{Q}\mathbf{v}),\quad \forall\,\mathbf{u},\mathbf{v}\in\mathcal{V},\ \forall\,\mathbf{Q}\in\mathrm{Orth}(\mathbf{e}), \tag{4.2}$$

where $\mathrm{Orth}(\mathbf{e})$ denotes the continuous group of all rotations about the span of $\mathbf{e}$.

On using a representation if type (4.1) for both $\widetilde{\mathbf{h}}$ and $\widetilde{\mathbf{q}}$:

$$\begin{aligned}\widetilde{\mathbf{h}}(\mathbf{u},\mathbf{v})&=h_1\,\mathbf{u}+h_2\,\mathbf{v}+h_3\,\mathbf{e},\quad h_i=\widetilde{h}_i(|\mathbf{u}|,|\mathbf{v}|,\mathbf{u}\cdot\mathbf{v},\mathbf{u}\cdot\mathbf{e},\mathbf{v}\cdot\mathbf{e})\;(i=1,2,3),\\ \widetilde{\mathbf{q}}(\mathbf{u},\mathbf{v})&=q_1\,\mathbf{u}+q_2\,\mathbf{v}+q_3\,\mathbf{e},\quad h_i=\widetilde{q}_i(|\mathbf{u}|,|\mathbf{v}|,\mathbf{u}\cdot\mathbf{v},\mathbf{u}\cdot\mathbf{e},\mathbf{v}\cdot\mathbf{e})\;(i=1,2,3),\end{aligned} \tag{4.3}$$

whence

$$\mathbf{h}-\lambda\mathbf{q}=(h_1-\lambda\,q_1)\mathbf{u}+(h_2-\lambda\,q_2)\mathbf{v}+(h_3-\lambda\,q_3)\mathbf{e}, \tag{4.4}$$

with the following provisional assumption as to the multiplier λ:

$$\lambda=\overline{\lambda}(|\mathbf{u}|,|\mathbf{v}|,\mathbf{u}\cdot\mathbf{v},\mathbf{u}\cdot\mathbf{e},\mathbf{v}\cdot\mathbf{e}). \tag{4.5}$$

Now, condition $(3.3)_1$ becomes:

$$\begin{aligned}&\big(h_1-\lambda q_1\big)\mathbf{I}+\big(\partial_1h_1-\lambda\partial_1q_1\big)|\mathbf{u}|^{-1}\,\mathbf{u}\otimes\mathbf{u}+\big(\partial_3h_2-\lambda\partial_3q_2\big)\,\mathbf{v}\otimes\mathbf{v}+\\&\big(\partial_4h_3-\lambda\partial_4q_3\big)\,\mathbf{e}\otimes\mathbf{e}+\\&\Big(\partial_3h_1+\partial_1h_2|\mathbf{u}|^{-1}-\lambda\big(\partial_3q_1+\partial_1q_2|\mathbf{u}|^{-1}\big)\Big)\,\mathrm{sym}\big(\mathbf{u}\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{u}\big)+\\&\Big(\partial_4h_1+\partial_1h_3|\mathbf{u}|^{-1}-\lambda\big(\partial_4q_1+\partial_1q_3|\mathbf{u}|^{-1}\big)\Big)\,\mathrm{sym}\big(\mathbf{u}\otimes\mathbf{e}+\mathbf{e}\otimes\mathbf{u}\big)+\\&\Big(\partial_4h_2+\partial_3h_3-\lambda\big(\partial_4q_2+\partial_3q_3\big)\Big)\,\mathrm{sym}\big(\mathbf{v}\otimes\mathbf{e}+\mathbf{e}\otimes\mathbf{v}\big)=\mathbf{0},\end{aligned} \tag{4.6}$$

for all $\mathbf{u}$ and $\mathbf{v}$; condition $(3.3)_2$ yields a completely analogous identity. By applying Lemma 2 in Appendix B to each of these two identities, we recover relations (3.9) and (3.10), as well as the following list of additional relations:

$$\begin{aligned}
&\partial_4 h_3 - \lambda\,\partial_4 q_3 = 0,\\
&\partial_4 h_1 - \lambda\,\partial_4 q_1 + (\partial_1 h_3 - \lambda\,\partial_1 q_3)|\mathbf{u}|^{-1} = 0,\\
&\partial_4 h_2 - \lambda\,\partial_4 q_2 + \partial_3 h_3 - \lambda\,\partial_3 q_3 = 0.
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&\partial_5 h_3 - \lambda\,\partial_5 q_3 = 0,\\
&\partial_5 h_2 - \lambda\,\partial_5 q_2 + (\partial_2 h_3 - \lambda\,\partial_2 q_3)|\mathbf{v}|^{-1} = 0,\\
&\partial_5 h_1 - \lambda\,\partial_5 q_1 + \partial_3 h_3 - \lambda\,\partial_3 q_3 = 0.
\end{aligned} \tag{4.8}$$

Now, we have shown that (3.9) and (3.10) imply that λ cannot depend on the first three arguments in the preliminary representation (4.5); moreover, due to $(3.9)_1$ and $(3.10)_1$, we have the following rearrangements of, respectively, conditions $(4.7)_2$ and $(4.8)_2$ and conditions $(4.7)_3$ and $(4.8)_3$:

$$\begin{aligned}
&(\partial_4\lambda) q_1 + \partial_1 (h_3 - \lambda\, q_3)|\mathbf{u}|^{-1} = 0,\\
&(\partial_5\lambda) q_2 + \partial_2 (h_3 - \lambda\, q_3)|\mathbf{v}|^{-1} = 0,
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
&(\partial_4\lambda) q_2 + \partial_3 h_3 - \lambda\,\partial_3 q_3 = 0,\\
&(\partial_5\lambda) q_1 + \partial_3 h_3 - \lambda\,\partial_3 q_3 = 0.
\end{aligned} \tag{4.10}$$

The last two relations imply that

$$(\partial_4\lambda) q_2 = (\partial_5\lambda) q_1,$$

a condition which poses no restrictions on the choice of $\mathbf{q}$ if and only if λ is independent of the fourth and fifth argument in (4.5). But, if λ may depend only on α and $\dot\alpha$, conditions (4.10) coalesce and, together with $(4.7)_1$, $(4.8)_1$, and (4.9), imply that $(h_3 - \lambda\, q_3)$ may also depend only on α and $\dot\alpha$.

All in all, we have that

$$h_1 - \lambda\, q_1 = 0, \quad h_2 - \lambda\, q_2 = 0, \quad h_3 - \lambda\, q_3 = f \quad \text{with} \quad f = \widetilde{f}(\alpha,\dot\alpha),\ \lambda = \widetilde{\lambda}(\alpha,\dot\alpha). \tag{4.11}$$

Accordingly, *an influx discrepancy in the direction of the transverse isotropy remains*:

$$\mathbf{h} - \lambda\,\mathbf{q} = (h_3 - \lambda\, q_3)\mathbf{e} = f\,\mathbf{e}, \tag{4.12}$$

where

$$\begin{aligned}
&\widetilde{h}_3(\alpha,\dot\alpha,|\nabla\alpha|,|\nabla\dot\alpha|,\nabla\alpha\cdot\nabla\dot\alpha,\partial_{\mathbf{e}}\alpha,\partial_{\mathbf{e}}\dot\alpha) =\\
&\qquad\qquad \widetilde{\lambda}(\alpha,\dot\alpha)\,\widetilde{q}_3(\alpha,\dot\alpha,|\nabla\alpha|,|\nabla\dot\alpha|,\nabla\alpha\cdot\nabla\dot\alpha,\partial_{\mathbf{e}}\alpha,\partial_{\mathbf{e}}\dot\alpha) + \widetilde{f}(\alpha,\dot\alpha).
\end{aligned}$$

5. Conclusions

The classical assumption of proportionality between entropy and energy influxes via the temperature has been accepted even in the case of non-classical continuum

theories, such as Green–Naghdi's, in spite of the objections raised by Müller (1971) and Liu (1972, 1973) and of the fact that it has been found inappropriate in several circumstances (for instance, when one deals with thermodynamics of diffusion (Liu 1973, Liu 2009)) and inconsistent with the kinetic theory of ideal gases (see (Liu 1973, Liu 2009) and references therein). As this assumption bears directly on the mathematical description of heat propagation, we have focused on rigid heat conductors within the framework of Green–Naghdi's Type III theory, and demonstrated its exclusively constitutive nature by showing that:
(i) when heat conduction is isotropic (that is, in Green–Naghdi's theory, when energy and entropy influxes are assumed to depend isotropically on the gradients of both the thermal displacement and its time rate), then entropy and energy influxes must be proprortial, via a multiplier that needs not be the temperature, although it may depend on it;
(ii) when heat conduction is transversely isotropic, then proportionality cannot be postulated in general, because an additional entropy flux along the direction of transverse isotropy is possible.

We believe that these results carry on without any substantial change for material classes more general than that of rigid conductors. What postulation about entropy and energy influxes fits one or another non-isotropic type of heat propagation is an issue that remains open.

Appendix A. Representation of an isotropic vector function of two vector arguments

Here we derive a representation formula for isotropic vector-valued mappings of two vector arguments; although the result is well known (Wang 1970, Wang 1971), our proof is, to our knowledge, new. As a premiss, we recall that a scalar-valued function f of two vectorial arguments is *isotropic*, i.e., such that

$$f(\mathbf{u},\mathbf{v}) = f(\mathbf{Qu},\mathbf{Qv}), \quad \forall \mathbf{u},\mathbf{v} \in \mathcal{V},\ \forall \mathbf{Q} \in \mathrm{Orth},$$

where $\mathcal{V}$ is a suitable vector space and Orth the full orthogonal group, if and only if

$$f(\mathbf{u},\mathbf{v}) = \varphi(|\mathbf{u}|,|\mathbf{v}|,\mathbf{u}\cdot\mathbf{v}). \tag{5.1}$$

LEMMA 1. *Let* $\widehat{\mathbf{f}}$ *be an* isotropic *mapping of* $\mathcal{V}$ *into itself, i.e., such that*

$$\mathbf{Q}\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v}) = \widehat{\mathbf{f}}(\mathbf{Qu},\mathbf{Qv}), \quad \forall \mathbf{u},\mathbf{v} \in \mathcal{V},\ \forall \mathbf{Q} \in \mathrm{Orth}. \tag{5.2}$$

Then, $\widehat{\mathbf{f}}$ *has the following representation:*

$$\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v}) = \varphi_1(|\mathbf{u}|,|\mathbf{v}|,\mathbf{u}\cdot\mathbf{v})\,\mathbf{u} + \varphi_2(|\mathbf{u}|,|\mathbf{v}|,\mathbf{u}\cdot\mathbf{v})\,\mathbf{v}. \tag{5.3}$$

Proof. Write (5.2) for $\mathbf{Q}$, a point of a smooth curve through $\mathbf{I}$, on the manifold Orth:

$$t \mapsto \mathbf{Q}(t), \qquad \mathbf{Q}(t_0) = \mathbf{I}, \qquad \dot{\mathbf{Q}}(t_0) =: \mathbf{W} \in \mathrm{Skw}, \tag{5.4}$$

and differentiate at $t=t_0$, so as to get:

$$\mathbf{W}\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v}) = \widehat{\mathbf{F}}_1(\mathbf{u},\mathbf{v})\mathbf{W}\mathbf{u} + \widehat{\mathbf{F}}_2(\mathbf{u},\mathbf{v})\mathbf{W}\mathbf{v},$$
$$\widehat{\mathbf{F}}_1(\mathbf{u},\mathbf{v}) := \partial_1\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v}),\ \widehat{\mathbf{F}}_2(\mathbf{u},\mathbf{v}) := \partial_2\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v}), \quad \forall \mathbf{u},\mathbf{v}\in\mathcal{V},\ \forall \mathbf{W}\in \mathrm{Skw}. \tag{5.5}$$

This condition is equivalent to

$$\mathbf{W}\cdot\big(\mathbf{a}\otimes\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v}) - \widehat{\mathbf{F}}_1^T(\mathbf{u},\mathbf{v})\mathbf{a}\otimes\mathbf{u} - \widehat{\mathbf{F}}_2^T(\mathbf{u},\mathbf{v})\mathbf{a}\otimes\mathbf{v}\big) = 0, \tag{5.6}$$

$\forall \mathbf{u},\mathbf{v},\mathbf{a}\in\mathcal{V}$, $\forall \mathbf{W}\in\mathrm{Skw}$, and then

$$\mathbf{a}\times\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v}) - \widehat{\mathbf{F}}_1^T(\mathbf{u},\mathbf{v})\mathbf{a}\times\mathbf{u} - \widehat{\mathbf{F}}_2^T(\mathbf{u},\mathbf{v})\mathbf{a}\times\mathbf{v} = \mathbf{0}, \quad \forall \mathbf{u},\mathbf{v},\mathbf{a}\in\mathcal{V}. \tag{5.7}$$

Now, take the vector product with $\mathbf{w}:=\mathbf{u}\times\mathbf{v}$ and use the identity $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$ so as to obtain:

$$(\mathbf{a}\cdot\mathbf{w})\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v}) - \big(\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v})\cdot\mathbf{w}\big)\mathbf{a} = \big(\mathbf{a}\cdot\widehat{\mathbf{F}}_1(\mathbf{u},\mathbf{v})\mathbf{w}\big)\mathbf{u} + \big(\mathbf{a}\cdot\widehat{\mathbf{F}}_2(\mathbf{u},\mathbf{v})\mathbf{w}\big)\mathbf{v} \tag{5.8}$$

$\forall \mathbf{u},\mathbf{v},\mathbf{a}\in\mathcal{V}$. Since

$$\begin{aligned}\big(\mathbf{a}\cdot\widehat{\mathbf{F}}_1(\mathbf{u},\mathbf{v})\mathbf{w}\big)\mathbf{u} &= \big(\mathbf{u}\otimes\widehat{\mathbf{F}}_1(\mathbf{u},\mathbf{v})\mathbf{w}\big)\mathbf{a} = \big(\mathbf{F}_1^T(\mathbf{u},\mathbf{v})\mathbf{u}\otimes\mathbf{w}\big)\mathbf{a},\\ \big(\mathbf{a}\cdot\widehat{\mathbf{F}}_2(\mathbf{u},\mathbf{v})\mathbf{w}\big)\mathbf{v} &= \big(\mathbf{v}\otimes\widehat{\mathbf{F}}_2(\mathbf{u},\mathbf{v})\mathbf{w}\big)\mathbf{a} = \big(\widehat{\mathbf{F}}_2^T(\mathbf{u},\mathbf{v})\mathbf{v}\otimes\mathbf{w}\big)\mathbf{a},\end{aligned} \tag{5.9}$$

we have that

$$(\mathbf{a}\cdot\mathbf{w})\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v}) - \big(\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v})\cdot\mathbf{w}\big)\mathbf{a} = \big(\widehat{\mathbf{F}}_1^T(\mathbf{u},\mathbf{v})\mathbf{u}\otimes\mathbf{w} + \widehat{\mathbf{F}}_2^T(\mathbf{u},\mathbf{v})\mathbf{v}\otimes\mathbf{w}\big)\mathbf{a}, \tag{5.10}$$

$\forall \mathbf{u},\mathbf{v},\mathbf{a}\in\mathcal{V}$. Given the arbitrariness of $\mathbf{a}$, it suffices to choose $\mathbf{a}\cdot\mathbf{w}=0$ to deduce that

$$\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v})\cdot\mathbf{w}=0, \quad \forall \mathbf{u},\mathbf{v}\in\mathcal{V} \quad \Leftrightarrow \quad \widehat{\mathbf{f}}(\mathbf{u},\mathbf{v}) = f_1(\mathbf{u},\mathbf{v})\mathbf{u} + f_2(\mathbf{u},\mathbf{v})\mathbf{v}. \tag{5.11}$$

To conclude, one observes that this provisional form of $\widehat{\mathbf{f}}(\mathbf{u},\mathbf{v})$ is compatible with (5.2) iff f_1 and f_2 are isotropic. Thus, an application of (5.1) suffices. ■

Appendix B. Two algebraic lemmas

LEMMA 2. *Let the algebraic equality*

$$\alpha\mathbf{I} + \beta\mathbf{u}\otimes\mathbf{u} + \gamma\mathbf{v}\otimes\mathbf{v} + \delta(\mathbf{u}\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}) = \mathbf{0} \tag{5.12}$$

hold for all $\mathbf{u},\mathbf{v}\in\mathcal{V}$. *Then,*

$$\alpha=\beta=\gamma=\delta=0. \tag{5.13}$$

LEMMA 3. *Let the algebraic equality*

$$\begin{aligned}&\alpha\mathbf{I} + \beta\mathbf{u}\otimes\mathbf{u} + \gamma\mathbf{v}\otimes\mathbf{v} + \delta(\mathbf{u}\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}) +\\ &\varphi\mathbf{e}\otimes\mathbf{e} + \chi(\mathbf{u}\otimes\mathbf{e} + \mathbf{e}\otimes\mathbf{u}) + \psi(\mathbf{v}\otimes\mathbf{e} + \mathbf{e}\otimes\mathbf{v}) = \mathbf{0}\end{aligned} \tag{5.14}$$

hold for all $\mathbf{u},\mathbf{v}\in\mathcal{V}$. *Then,*

$$\alpha=\beta=\gamma=\delta=\varphi=\chi=\psi=0. \tag{5.15}$$

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